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Quasideterminant multisoliton solutions of a supersymmetric chiral field model in two dimensions

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Abstract
The Darboux transformation of a supersymmetric principal chiral field model in two dimensions is studied. We obtain exact superfield multisoliton solutions of the model by means of iterated Darboux transformation and express them in terms of quasideterminants.

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1. Introduction

In previous work [1] we investigated Darboux transformation of a principal chiral field model in two dimensions. We obtained the multisoliton solutions of the chiral model using the Darboux matrix approach. The matrix solutions of the chiral model and those of its associated Lax pair were shown to be expressed in terms of quasideterminants. The quasideterminant solutions were also shown to be related to the solutions obtained by Zakharov and Mikhailov using the dressing method based on the matrix Riemann–Hilbert problem.

In this paper we extend these earlier results for the case of a supersymmetric chiral field model in two dimensions. The classical integrability of the model is studied extensively, see e.g. [2–9]. We study Darboux transformation by introducing a superfield Darboux matrix for the supersymmetric chiral model and obtain superfield multisoliton solutions using the iterated Darboux transformations. The superfield multisoliton solutions of the supersymmetric chiral field were obtained by Mikhailov [2] by using the generalization of the method of Zakharov and Mikhailov [15]. We use a related method of Darboux transformation by defining a superfield Darboux matrix. We show that the superfield multisoliton solutions of the supersymmetric chiral field model can be expressed in terms of quasideterminants. The main results of this paper are as follows: the superfield generalization of the soliton solution generating method of Darboux transformation in terms of the superfield Darboux matrix, the construction of superfield multisoliton solutions in terms of quasideterminants, explicit soliton solutions for the model based on the Lie group $SU(2)$. The K times iteration of Darboux transformation to obtain quasideterminant *K*th conserved superfield currents and quasideterminant superfield *K* solitons of the model in terms of initial (seed) solution is also studied.

Following [2], we define a Grassmann even superfield $G(x^{\pm}, \theta^{\pm})$ with values in a Lie group $\mathcal{G}$. The superspace Lagrangian of the supersymmetric chiral field with target space $\mathcal{G}$ is then given by[1]

$$\mathcal{L}_{\mathrm{SCF}} = \tfrac{1}{2}\mathrm{Tr}(D_+G^{-1}D_-G) \tag{1.1}$$

and $G(x,\theta)G^{-1}(x,\theta) = 1 = G^{-1}(x,\theta)G(x,\theta)$. The superspace Lagrangian $\mathcal{L}_{\mathrm{SCF}}$ is invariant under the global transformation $\mathcal{G}_L \times \mathcal{G}_R : G(x^{\pm},\theta^{\pm}) = \mathcal{U}G(x^{\pm},\theta^{\pm})\mathcal{V}^{-1}$, where $\mathcal{U}$ and $\mathcal{V}$ are $\mathcal{G}_L$ and $\mathcal{G}_R$ valued Grassmann even matrix superfields, respectively. The Noether conserved superfield currents associated with $\mathcal{G}_R$ are $J_{\pm} = -\mathrm{i}G^{-1}D_{\pm}G$ which are Grassmann odd superfields and are Lie algebra $\mathfrak{g}$ valued, i.e. $J_{\pm} = J^a_{\pm}T^a$, where the anti-Hermitian generators $\{T^a, a = 1, 2, \ldots, \dim\mathfrak{g}\}$ of the Lie algebra $\mathfrak{g}$ obey $[T^a, T^b] = f^{abc}T^c$ and $\mathrm{Tr}\,(T^aT^b) = \delta^{ab}$. For any $X \in \mathfrak{g}$, $X = X^aT^a$. The conserved superfield current corresponding to the $\mathcal{G}_L$ transformation is $-GJ_{\pm}G^{-1}$. The superfield equation of motion of the supersymmetric chiral field is the superfield conservation equation and the superfield zero-curvature condition:

$$D_+J_- - D_-J_+ = 0, \tag{1.2}$$

$$D_-J_+ + D_+J_- + \mathrm{i}\{J_+, J_-\} = 0. \tag{1.3}$$

We can expand the matrix superfield $G(x^{\pm}, \theta^{\pm})$ in terms of bosonic and fermionic component fields as

$$G(x,\theta) = g(x)(1 + \mathrm{i}\theta^+\psi_+(x) + \mathrm{i}\theta^-\psi_-(x) + \mathrm{i}\theta^+\theta^-F(x)), \tag{1.4}$$

where $\psi_{\pm}$ are the Majorana spinors such that $\psi^R_{\pm} = g^{-1}\psi^L_{\pm}g$, and $F(x)$ is the auxiliary field, with an algebraic equation of motion[2]. The Majorana spinors $\psi_{\pm}(x)$ take values in the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. The action of the symmetry $\mathcal{G}_L \times \mathcal{G}_R$ on component fields is given by

$$g \longmapsto UgV^{-1}, \qquad \psi^R_{\pm} \longmapsto V\psi^R_{\pm}V^{-1}, \qquad \psi^L_{\pm} \longmapsto U\psi^L_{\pm}U^{-1}, \tag{1.5}$$

where U and V are the leading bosonic components of the matrix superfields $\mathcal{U}$ and $\mathcal{V}$ respectively, i.e. the fermions transform under $\mathcal{G}_R$ or $\mathcal{G}_L$.

After the elimination of the auxiliary field from the expression of $G(x,\theta)$, the component Lagrangian finally becomes

$$\begin{aligned}\mathcal{L} = -\frac{1}{2}\bigg(&g^{-1}\partial_+gg^{-1}\partial_-g + \frac{\mathrm{i}}{2}\psi_+\partial_-\psi_+ + \frac{\mathrm{i}}{2}\psi_+[g^{-1}\partial_-g, \psi_+] \\ &+ \frac{\mathrm{i}}{2}\psi_-\partial_+\psi_- + \frac{\mathrm{i}}{2}\psi_-[g^{-1}\partial_+g, \psi_-] + \frac{1}{2}\psi_+^2\psi_-^2\bigg).\end{aligned} \tag{1.6}$$

[1] In our notation the superspace is defined with coordinates $(x^{\pm}, \theta^{\pm})$, where $x^{\pm}$ are the commuting light-cone coordinates of the two-dimensional Minkowski space $\mathbb{R}^{1,1}$ with metric $\eta_{\mu\nu} = \mathrm{diag}(1, -1)$ and $\theta^{\pm}$ are anticommuting fermionic coordinates and are real Grassmann numbers. The light-cone coordinates $x^{\pm}$ are related to the orthonormal coordinates by $x^{\pm} = \frac{1}{2}(t \pm x)$ with the derivatives $\partial_{\pm} = \partial_t \pm \partial_x$. The $N = 1$ supercharges are $Q_{\pm} = \frac{\partial}{\partial\theta^{\pm}} + \mathrm{i}\theta^{\pm}\partial_{\pm}$, and the superspace covariant derivatives are $D_{\pm} = \frac{\partial}{\partial\theta^{\pm}} - \mathrm{i}\theta^{\pm}\partial_{\pm}$, which obey $\{Q_{\pm}, Q_{\pm}\} = \mathrm{i}2\partial_{\pm}$, $\{D_{\pm}, D_{\pm}\} = -\mathrm{i}2\partial_{\pm}$, with all other anti-commutators vanishing. Under the Lorentz transformation the coordinates of superspace and their derivatives transform as $x^{\pm} \mapsto \mathrm{e}^{\pm\eta}x^{\pm}$, $\partial_{\pm} \mapsto \mathrm{e}^{\mp\eta}\partial_{\pm}$, $\theta_{\pm} \mapsto \mathrm{e}^{\pm\frac{1}{2}\eta}\theta_{\pm}$ and $D_{\pm} \mapsto \mathrm{e}^{\mp\frac{1}{2}\eta}D_{\pm}$, where η is the rapidity of the Lorentz boost.

[2] Note that our conventions are such that the γ-matrices $\gamma_0 = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0\end{pmatrix}$, $\gamma_1 = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0\end{pmatrix}$ satisfy $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$. The Dirac spinor is $\psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}$, where $\psi_{\pm}$ are chiral spinors and our assumption is that $\psi_{\pm}$ are real (Majorana) and the Lorentz behavior of these Majorana spinors is $\psi_{\pm} \mapsto \mathrm{e}^{\mp\frac{1}{2}\eta}\psi_{\pm}$.

Using Euler–Lagrange equations, we can directly find the component equations of motion for the supersymmetric chiral field. From the definition of superfield current $J_\pm$, we write its component expansion as

$$J_\pm = \psi_\pm + \theta^\pm j_\pm - \frac{\mathrm{i}}{2}\theta^\mp\{\psi_+, \psi_-\} - \mathrm{i}\theta^+\theta^-\left(\partial_\pm\psi_\mp - [j_\pm, \psi_\mp] - \frac{\mathrm{i}}{2}\left[\psi_\pm^2, \psi_\mp\right]\right), \tag{1.7}$$

where the component bosonic conserved current is $j_\pm = -\left(g^{-1}\partial_\pm g + \mathrm{i}\psi_\pm^2\right)$. Substituting these into the superspace equations of motion, collecting terms and writing $h_\pm = \psi_\pm^2 \Leftrightarrow h_\pm^a = \frac{1}{2}f^{abc}\psi_\pm^b\psi_\pm^c$, we get the equations of motion for fermionic fields $\psi_\pm$ and bosonic fields $j_\pm$ of the supersymmetric chiral field model:

$$\partial_\pm\psi_\mp - \frac{1}{2}[j_\pm, \psi_\mp] - \frac{\mathrm{i}}{4}[h_\pm, \psi_\mp] = 0, \tag{1.8}$$

$$\partial_- j_+ + \partial_+ j_- = 0. \tag{1.9}$$

We use the fermion equations of motion to get the following equations:

$$\partial_- j_+ - \partial_+ j_- + [j_+, j_-] = \mathrm{i}(\partial_- h_+ - \partial_+ h_-), \tag{1.10}$$

$$\partial_\mp h_\pm + \frac{1}{2}\left[h_\pm, j_\mp + \frac{\mathrm{i}}{2}h_\mp\right] = 0, \tag{1.11}$$

$$\partial_- h_+ + \partial_+ h_- = 0. \tag{1.12}$$

Equations (1.9) and (1.12) show the conservation of bosonic currents $j_\pm$ and $h_\pm$, respectively.

The supersymmetric chiral field equations (1.2)–(1.3) can be written as the compatibility condition of the following superfield Lax pair:

$$D_\pm\mathcal{V}(\lambda) = -\frac{\mathrm{i}}{1\mp\lambda}J_\pm\mathcal{V}(\lambda), \tag{1.13}$$

where λ is a real (or complex) parameter. The component expansion of the Lax pair (1.13) gives

$$\partial_\pm V(x^\pm;\lambda) = \frac{1}{1\mp\lambda}j_\pm V(x^\pm;\lambda) + \mathrm{i}\left(\frac{1}{1\mp\lambda}\right)^2 h_\pm V(x^\pm;\lambda), \tag{1.14}$$

where $V(x^\pm,\lambda)$ is the leading purely bosonic component of the superfield $\mathcal{V}(x^\pm,\theta^\pm,\lambda)$. In this case, the compatibility condition of (1.14) gives equations (1.8)–(1.9). Therefore, the partial differential equations (1.2)–(1.3) have the Lax pair (1.13), while equations (1.8)–(1.12) have the Lax pair (1.14).

We will use the notion of quasideterminants in order to express the superfield multisoliton solutions of the supersymmetric chiral field model. The quasideterminants were introduced by Gelfand and Retakh [28–32]. Let X be an $N\times N$ matrix over a ring R (noncommutative, in general). For any $1\leqslant i, j\leqslant N$, let $r_i^{\ j}$ be a row matrix obtained by removing the jth entry of X from the ith row. Similarly, $c_j^{\ i}$ is the column matrix containing the jth column of X without the ith entry. There exist N^2 quasideterminants denoted by $|X|_{ij}$ for $i, j = 1,\ldots,N$ and are defined by

$$|X|_{ij} = \begin{vmatrix} X^{ij} & c_j^{\ i} \\ r_i^{\ j} & \boxed{x_{ij}} \end{vmatrix} = x_{ij} - r_i^{\ j}(X^{ij})^{-1}c_j^{\ i}. \tag{1.15}$$

In the block form, the quasideterminant of the $K\times K$ matrix expanded about the $N\times N$ matrix D is defined by

$$\begin{vmatrix} A & B \\ C & \boxed{D} \end{vmatrix} = D - CA^{-1}B. \tag{1.16}$$

For various properties and applications of quasideterminants in the theory of integrable systems, see e.g. [28–40].

2. Darboux transformation on superfields

In order to construct multisoliton solutions of the supersymmetric chiral field model, we define Darboux transformation on the superfields that provides a natural generalization of Darboux transformation to the supersymmetric case. In the present case, the Darboux transformation is defined by an $N \times N$ Grassmann even superfield matrix $\mathcal{D}(x^{\pm}, \theta^{\pm}, \lambda)$, called the superfield Darboux matrix (for a discussion on the method of the Darboux matrix see [10–21]. The superfield Darboux matrix connects the two superfield solutions of the Lax pair (1.13). Let us define the superfield matrix $\mathcal{D}(x^{\pm}, \theta^{\pm}, \lambda)$, such that the superfield $\mathcal{V}$ of the Lax pair (1.13) transforms as

$$\tilde{\mathcal{V}} = \mathcal{D}(\lambda)\mathcal{V}, \tag{2.1}$$

with suitable $\tilde{J}_{\pm}$ satisfying equations (1.2)–(1.3). That is the new solutions $\tilde{J}_{\pm}, \tilde{\mathcal{V}}$ are defined in such a way that the superfield Lax pair (1.13) is covariant under the Darboux transformation. That is, the transformed superfields satisfy the same Lax pair

$$D_{\pm}\tilde{\mathcal{V}} = -\frac{\mathrm{i}}{1 \mp \lambda}\tilde{J}_{\pm}\tilde{\mathcal{V}}. \tag{2.2}$$

By substituting equation (2.1) into equation (2.2), we get the following condition on the superfield Darboux matrix $\mathcal{D}(\lambda)$:

$$D_{\pm}\mathcal{D}(\lambda)\mathcal{V}(\lambda) + \mathcal{D}(\lambda)\frac{1}{1 \mp \lambda}(-\mathrm{i}J_{\pm})\mathcal{V}(\lambda) = \frac{1}{1 \mp \lambda}(-\mathrm{i}\widetilde{J}_{\pm})\mathcal{D}(\lambda)\mathcal{V}(\lambda). \tag{2.3}$$

For our system, we make the following ansatz for the superfield Darboux matrix:

$$\mathcal{D}(x^{+}, x^{-}, \lambda) = \lambda I - \mathcal{S}(x^{+}, x^{-}), \tag{2.4}$$

where $\mathcal{S}(x^{+}, x^{-})$ is some $N \times N$ matrix superfield to be determined and I is an $N \times N$ identity matrix. Note that we consider here the superfield Darboux matrix of degree 1 which is linear in λ. Therefore, to construct the superfield Darboux matrix $\mathcal{D}(x^{+}, x^{-}, \lambda)$, it is only necessary to determine the matrix $\mathcal{S}(x^{+}, x^{-})$. Also note that our construction of the superfield Darboux matrix is based on the approach initiated in [1] and [20], for the bosonic chiral model. In comparison, the construction of the superfield Darboux matrix in [2] is based on the dressing method of Zakharov and Mikhailov [15]. The important point, however, is to express the superfield matrix $\mathcal{S}$ in terms of solutions of the linear problem.

Now substituting (2.1) in equation (2.2) and using (1.13), we get the following Darboux transformation for the superfield current components:

$$\tilde{J}_{\pm} = J_{\pm} \pm \mathrm{i}\, D_{\pm}\mathcal{S}, \tag{2.5}$$

and the matrix superfield $\mathcal{S}$ is subjected to satisfy the following equations:

$$D_{\pm}\mathcal{S}(I \mp \mathcal{S}) = -\mathrm{i}[J_{\pm}, \mathcal{S}]. \tag{2.6}$$

These new (Darboux transformed) superfield currents are also conserved and obey the superfield zero-curvature condition, i.e.

$$D_{+}\tilde{J}_{-} - D_{-}\tilde{J}_{+} = 0 \tag{2.7}$$

$$D_{-}\tilde{J}_{+} + D_{+}\tilde{J}_{-} + \mathrm{i}\{\tilde{J}_{+}, \tilde{J}_{-}\} = 0. \tag{2.8}$$

In the supersymmetric chiral field model, there are Grassmann even superfields such as $D_\pm J_\pm$ and $J_\pm^2$ which are relevant especially when we expand the superfields in terms of component fields. The bosonic superfield conserved currents $D_\pm J_\pm$ and $J_\pm^2$ transform as

$$D_\pm \tilde{J}_\pm = D_\pm J_\pm \pm \partial_\pm \mathcal{S}, \qquad \tilde{J}_\pm^2 = (I \mp \mathcal{S}) J_\pm^2 (I \mp \mathcal{S})^{-1}, \tag{2.9}$$

and the conditions on $\mathcal{S}$ are now written as

$$\partial_\pm \mathcal{S}(I \mp \mathcal{S}) = \pm D_\pm J_\pm \mathcal{S} \pm \mathcal{S} D_\pm J_\pm - \mathrm{i} J_\pm^2 + \mathrm{i}(I \mp \mathcal{S}) J_\pm^2 (I \mp \mathcal{S})^{-1}. \tag{2.10}$$

For a non-trivial solution of equation (2.6) or (2.10), we proceed as follows.

Now following the strategy of the non-supersymmetric model [1], we construct a matrix superfield from the particular column superfield solutions of the superfield Lax pair at particular values of the spectral parameter. Let $\lambda_1, \ldots, \lambda_N$ be N distinct real (or complex) constant parameters and $\lambda_i \neq \pm 1; i = 1, 2, \ldots, N$. Let us also define N constant column vectors $|1\rangle, |2\rangle, \ldots, |N\rangle$ such that

$$\mathcal{M} = (\mathcal{V}(\lambda_1)\,|1\rangle, \ldots, \mathcal{V}(\lambda_N)\,|N\rangle) = (|\Theta_1\rangle, \ldots, |\Theta_N\rangle) \tag{2.11}$$

be an invertible $N \times N$ matrix superfield. Each column $|\Theta_i\rangle = \mathcal{V}(\lambda_i)\,|i\rangle$ in $\mathcal{M}$ is a column solution of the superfield Lax pair (1.13) when $\lambda = \lambda_i$, i.e. it satisfies

$$D_\pm |\Theta_i\rangle = -\frac{\mathrm{i}}{1 \mp \lambda_i} J_\pm |\Theta_i\rangle, \tag{2.12}$$

and $i = 1, 2, \ldots, N$. Assuming $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, equation (2.12) can be written in matrix form as

$$D_\pm \mathcal{M} = -\mathrm{i} J_\pm \mathcal{M}(I \mp \Lambda)^{-1}. \tag{2.13}$$

The matrix superfield $\mathcal{S}$ is now chosen such that when substituted in equation (2.4), it gives the Darboux transformation. Now we make the following choice of the matrix superfield $\mathcal{S}$:

$$\mathcal{S} = \mathcal{M} \Lambda \mathcal{M}^{-1}. \tag{2.14}$$

The matrix superfield $\mathcal{S}$ is expressed in terms of the solutions of the linear problem which allows the construction of new solutions by means of the superfield Darboux matrix $\mathcal{D}$. It can be easily checked that the choice (2.14) satisfies condition (2.6).

Since we have assumed $\mathcal{M}$ to be invertible, therefore, we require that $\det\mathcal{M} \neq 0$. At this stage, we conclude that if the set of superfields $(J_\pm, \mathcal{V})$ is a solution of the Lax pair (1.13) and the matrix superfield $\mathcal{S}(x^\pm, \theta^\pm)$ is defined by (2.14), then the set of matrix superfields $(\tilde{J}_\pm, \tilde{\mathcal{V}})$ defined by (2.5), (2.1) via Darboux transformation (2.4) is also a solution of the same Lax pair. This establishes the covariance of the superfield Lax pair (1.13) under the Darboux transformation (2.4).

Now if the target space of the supersymmetric chiral field model is the Lie group $U(N)$, then we also require for the new superfield solutions to take values in $U(N)$. We know that the Lie group $U(N)$ consists of all $N \times N$ matrices G such that $G^\dagger = G^{-1}$. An arbitrary matrix X belongs to the Lie algebra $u(N)$ of the Lie group $U(N)$ if and only if $X^\dagger = -X$. Since the superfield currents $J_\pm$ are $u(N)$ valued, we require that the new superfield currents $\tilde{J}_\pm$ obtained by Darboux transformation must be $u(N)$ valued, i.e. they must be anti-Hermitian. This leads to the following condition on the matrix superfield $\mathcal{S}(x^\pm, \theta^\pm)$:

$$D_\pm(\mathcal{S} + \mathcal{S}^\dagger) = 0. \tag{2.15}$$

For the matrix superfield $\mathcal{S}$ to satisfy (2.15), we proceed by taking specific values of parameters λ_i. Let $\lambda_i = \mu$ $(i = 1, 2, \ldots, N)$ be a non-zero complex number; then following [1] we get the following conditions on the superfield matrix $\mathcal{S}$:

$$(\mathcal{S}^\dagger + \mathcal{S}) = (\mu + \bar{\mu})I, \tag{2.16}$$

$$\mathcal{S}^\dagger \mathcal{S} = \mu\bar{\mu}. \tag{2.17}$$

It is important to note that our solution of the linear system (1.13) is defined globally and satisfies the reality condition

$$\mathcal{V}^\dagger(\bar{\lambda})\mathcal{V}(\lambda) \in \{I\}. \tag{2.18}$$

To obtain global defined transformed solutions, the Darboux transformation must preserve this reality condition, i.e.

$$\tilde{\mathcal{V}}^\dagger(\bar{\lambda})\tilde{\mathcal{V}}(\lambda) \in \{I\}. \tag{2.19}$$

Using (2.1) and (2.4), also making use of (2.16) and (2.17), we see that

$$\tilde{\mathcal{V}}^\dagger(\bar{\lambda})\tilde{\mathcal{V}}(\lambda) = (\lambda^2 - \lambda(\mu + \bar{\mu}) + \mu\bar{\mu})\mathcal{V}^\dagger(\bar{\lambda})\mathcal{V}(\lambda) \in \{I\};$$

the transformed solution satisfies the reality condition.

3. Quasideterminant superfield multisoliton solutions

Now we express the superfield multisoliton solutions of the supersymmetric chiral field model in terms of quasideterminants of matrices over a noncommutative ring of matrices whose entries are Grassmann even superfields. We can write equation (2.1) in terms of quasideterminant as

$$\begin{aligned}\tilde{\mathcal{V}} &= D(\lambda)\mathcal{V} = (\lambda I - \mathcal{S})\mathcal{V},\\ &= (\lambda I - \mathcal{M}\Lambda\mathcal{M}^{-1})\mathcal{V} = \begin{vmatrix} \mathcal{M} & I \\ \mathcal{M}\Lambda & \boxed{\lambda I} \end{vmatrix}\mathcal{V}\end{aligned} \tag{3.1}$$

and the chiral superfield $\tilde{G}$ is obtained from equations (3.1), i.e.

$$\tilde{G} = \tilde{\mathcal{V}}(0) = -\mathcal{S}G = -(\mathcal{M}\Lambda\mathcal{M}^{-1}) = \begin{vmatrix} \mathcal{M} & I \\ \mathcal{M}\Lambda & \boxed{0} \end{vmatrix} G. \tag{3.2}$$

Similarly the conserved superfield currents $J_\pm$ are expressed as

$$\begin{aligned}\tilde{J}_\pm &= \mathcal{M}(I \mp \Lambda)\mathcal{M}^{-1}J_\pm\mathcal{M}(I \mp \Lambda)^{-1}\mathcal{M}^{-1},\\ &= \begin{vmatrix} \mathcal{M} & I \\ \mathcal{M}(I \mp \Lambda) & \boxed{O} \end{vmatrix} j_\pm \begin{vmatrix} \mathcal{M} & I \\ \mathcal{M}(I \mp \Lambda) & \boxed{O} \end{vmatrix}^{-1}.\end{aligned} \tag{3.3}$$

Using the notation $\mathcal{V}[1] = \mathcal{V}$, $G[1] = G$, $J_\pm[1] = J_\pm$ and $\mathcal{V}[2] = \tilde{\mathcal{V}}$, $G[2] = \tilde{G}$, $J_\pm[2] = \tilde{J}_\pm$, we write the K-fold Darboux transformation on the superfield $\mathcal{V}$ as

$$\begin{aligned}\mathcal{V}[K+1] &= \prod_{k=1}^{K}(\lambda I - \mathcal{S}[K-k+1])\,\mathcal{V} = \prod_{k=1}^{K}\begin{vmatrix} \mathcal{M}[K-k+1] & I \\ \mathcal{M}[K-k+1]\Lambda_{K-k+1} & \boxed{\lambda I} \end{vmatrix}\mathcal{V},\\ &= \lambda V[K] - \mathcal{M}[K]\Lambda_K\mathcal{M}[K]^{-1}\mathcal{V}[K],\\ &= \begin{vmatrix} \mathcal{M}_1 & \cdots & \mathcal{M}_K & I \\ \mathcal{M}_1\Lambda_1 & \cdots & \mathcal{M}_K\Lambda_K & \lambda I \\ \mathcal{M}_1\Lambda_1^2 & \cdots & \mathcal{M}_K\Lambda_K^2 & \lambda^2 I \\ \vdots & \cdots & \vdots & \vdots \\ \mathcal{M}_1\Lambda_1^K & \cdots & \mathcal{M}_K\Lambda_K^K & \boxed{\lambda^K I} \end{vmatrix}\mathcal{V},\end{aligned} \tag{3.4}$$

where for each $k = 1, 2, \ldots, K$, we have defined an invertible $N \times N$ matrix superfield $\mathcal{M}_k$ being the matrix superfield solution of the Lax pair (1.13) at $\Lambda = \Lambda_k$. The above results can be proved by induction using the properties of quasideterminants. Similar results have been proved in [1]. The multisoliton solution $G[K+1]$ of the chiral model can be readily obtained by taking $\lambda = 0$ in the expression of $\mathcal{V}[K+1]$, i.e.

$$G[K+1] = \prod_{k=1}^{K} (-1)^k\, \mathcal{S}[K-k+1]G = \prod_{k=1}^{K} \begin{vmatrix} \mathcal{M}[K-k+1] & I \\ \mathcal{M}[K-k+1]\Lambda_{K-k+1} & \boxed{O} \end{vmatrix} G,$$

$$= \begin{vmatrix} \mathcal{M}_1 & \mathcal{M}_2 & \cdots & \mathcal{M}_K & I \\ \mathcal{M}_1\Lambda_1 & \mathcal{M}_2\Lambda_2 & \cdots & \mathcal{M}_K\Lambda_K & O \\ \mathcal{M}_1\Lambda_1^2 & \mathcal{M}_2\Lambda_2^2 & \cdots & \mathcal{M}_K\Lambda_K^2 & O \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \mathcal{M}_1\Lambda_1^K & \mathcal{M}_2\Lambda_2^K & \cdots & \mathcal{M}_K\Lambda_K^K & \boxed{O} \end{vmatrix} G. \tag{3.5}$$

Similarly the K times iteration of the Darboux transformation gives the following expression of the conserved currents:

$$J_\pm[K+1] = (I \mp \mathcal{S}[K]) \cdots (I \mp \mathcal{S}[2])(I \mp \mathcal{S}[1]) J_\pm((I \mp \mathcal{S}[1])(I \mp \mathcal{S}[2]) \cdots (I \mp \mathcal{S}[K]))^{-1},$$

$$= \prod_{k=1}^{K} \begin{vmatrix} \mathcal{M}[K-k+1] & I \\ \mathcal{M}[K-k+1](I \mp \Lambda_{K-k+1}) & \boxed{O} \end{vmatrix} J_\pm \prod_{l=1}^{K} \begin{vmatrix} \mathcal{M}[l] & I \\ \mathcal{M}[l](I \mp \Lambda_l) & \boxed{O} \end{vmatrix}^{-1}$$

$$= \begin{vmatrix} \mathcal{M}_1 & \cdots & \mathcal{M}_K & I \\ \mathcal{M}_1(I \mp \Lambda_1) & \cdots & \mathcal{M}_K(I \mp \Lambda_K) & O \\ \mathcal{M}_1(I \mp \Lambda_1)^2 & \cdots & \mathcal{M}_K(I \mp \Lambda_K)^2 & O \\ \vdots & \cdots & \vdots & \vdots \\ \mathcal{M}_1(I \mp \Lambda_1)^K & \cdots & \mathcal{M}_K(I \mp \Lambda_K)^K & \boxed{O} \end{vmatrix}$$

$$\times J_\pm \begin{vmatrix} \mathcal{M}_1 & \cdots & \mathcal{M}_K & I \\ \mathcal{M}_1(I \mp \Lambda_1) & \cdots & \mathcal{M}_K(I \mp \Lambda_K) & O \\ \mathcal{M}_1(I \mp \Lambda_1)^2 & \cdots & \mathcal{M}_K(I \mp \Lambda_K)^2 & O \\ \vdots & \cdots & \vdots & \vdots \\ \mathcal{M}_1(I \mp \Lambda_1)^K & \cdots & \mathcal{M}_K(I \mp \Lambda_K)^K & \boxed{O} \end{vmatrix}^{-1}. \tag{3.6}$$

Equation (3.6) gives the required expressions of Kth conserved superfield current components of the supersymmetric chiral model expressed in terms of quasideterminants involving particular solutions of the superfield linear problem.

The bosonic superfield currents $D_\pm J_\pm$ and $J_\pm^2$ that satisfy equation (2.9) can be written in terms of quasideterminants as follows. The first set of equations (2.9) can be written as

$$\begin{aligned} D_\pm \tilde{J}_\pm &= (I \mp \mathcal{S}) D_\pm J_\pm (I \mp \mathcal{S})^{-1} - \mathrm{i} J_\pm^2 (I \mp \mathcal{S})^{-1} + (I \mp \mathcal{S}) \mathrm{i} J_\pm^2 (I \mp \mathcal{S})^{-2} \\ &= (I \mp \mathcal{S}) D_\pm J_\pm (I \mp \mathcal{S})^{-1} - (I \mp \mathcal{S}) \left[(I \mp \mathcal{S})^{-1}, \mathrm{i} J_\pm^2\right] (I \mp \mathcal{S})^{-1} \\ &= \mathcal{Q}_\pm[1] \left(D_\pm J_\pm + \left[\mathrm{i} J_\pm^2, \mathcal{Q}_\pm[1]^{-1}\right]\right) \mathcal{Q}_\pm[1]^{-1}, \end{aligned} \tag{3.7}$$

where the bracket [,] in the second term of (3.7) is the commutator and

$$\mathcal{Q}_\pm[1] = (I \mp \mathcal{S}[1]) = \begin{vmatrix} \mathcal{M} & I \\ \mathcal{M}(I \mp \Lambda) & \boxed{O} \end{vmatrix}. \tag{3.8}$$

Similarly the Darboux transformation of the superfield $J_\pm^2$ is given by

$$\tilde{J}_\pm^2 = (I \mp \mathcal{S}) J_\pm^2 (I \mp \mathcal{S})^{-1} = \mathcal{Q}_\pm[1] J_\pm^2 \mathcal{Q}_\pm^{-1}[1]. \tag{3.9}$$

The Kth iteration of bosonic superfield currents $D_\pm J_\pm$ and $J_\pm^2$ can be applied in a similar way and we get

$$D_\pm J_\pm[K+1] = \mathcal{Q}_\pm[K]\left(D_\pm J_\pm + \left[\mathrm{i}J_\pm^2, \sum_{k=1}^{K} \mathcal{Q}_\pm[k]^{-1}\,\mathcal{Q}[k-1]\right]\right)\mathcal{Q}_\pm[K]^{-1}, \tag{3.10}$$

$$J_\pm^2[K+1] = \mathcal{Q}_\pm[K] J_\pm^2 \mathcal{Q}_\pm[K]^{-1}, \tag{3.11}$$

where

$$\mathcal{Q}_\pm[K] = (I \mp \mathcal{S}[K])\cdots(I \mp \mathcal{S}[2])\,(I \mp \mathcal{S}[1]). \tag{3.12}$$

Expression (3.11) is similar to expression (3.6) and can be easily proved by induction (see [1]). Equation (3.10) is slightly different; therefore, we give some details of inductive proof. From equation (3.7), it is clear that expression (3.10) is true for $K = 1$. Now we show that $D_\pm J_\pm[K+2]$ is true, given that (3.10) is true. For this we consider

$$\begin{aligned} D_\pm J_\pm[K+2] &= D_\pm((I \mp \mathcal{S}[K+1])J_\pm[K+1](I \mp \mathcal{S}[K+1])^{-1}) \\ &= \mp D_\pm \mathcal{S}[K+1](J_\pm[K+1](I \mp \mathcal{S}[K+1])^{-1}) \\ &\quad + (I \mp \mathcal{S}[K+1])(D_\pm J_\pm[K+1])(I \mp \mathcal{S}[K+1])^{-1} \\ &\quad \mp (I \mp \mathcal{S}[K+1])J_\pm[K+1](I \mp \mathcal{S}[K+1])^{-1} D_\pm \mathcal{S}[K+1](I \mp \mathcal{S}[K+1])^{-1}. \end{aligned}$$

By using equation (3.10) in the second term and

$$D_\pm \mathcal{S}[K+1] = \mp \mathrm{i} J_\pm[K+2] \pm \mathrm{i} J_\pm[K+1],$$

in the first and third terms, we get

$$\begin{aligned} D_\pm J_\pm[K+2] &= (I \mp \mathcal{S}[K+1])\mathcal{Q}_\pm[K] D_\pm J_\pm \mathcal{Q}_\pm[K]^{-1}(I \mp \mathcal{S}[K+1])^{-1} \\ &\quad + (I \mp \mathcal{S}[K+1])\mathcal{Q}_\pm[K]\left[\mathrm{i}J_\pm^2, \sum_{k=1}^{K}\mathcal{Q}_\pm[k]^{-1}\mathcal{Q}_\pm[k-1]\right]\mathcal{Q}_\pm[K]^{-1}(I \mp \mathcal{S}[K+1])^{-1} \\ &\quad - \left(\mathrm{i}J_\pm^2[K+1] - \mathrm{i}J_\pm^2[K+2]\right)(I \mp \mathcal{S}[K+1])^{-1} \\ &= \mathcal{Q}_\pm[K+1]\left(D_\pm J_\pm + \left[\mathrm{i}J_\pm^2, \sum_{k=1}^{K+1}\mathcal{Q}_\pm[k]^{-1}\,\mathcal{Q}_\pm[\mathrm{i}-1]\right]\right)\mathcal{Q}[K+1]^{-1}. \end{aligned}$$

The quasideterminant expression for $\mathcal{Q}_\pm[K]$ is

$$\mathcal{Q}_\pm[K] = \begin{vmatrix} \mathcal{M}_1 & \cdots & \mathcal{M}_K & I \\ \mathcal{M}_1(I \mp \Lambda_1) & \cdots & \mathcal{M}_K(I \mp \Lambda_K) & O \\ \vdots & \cdots & \vdots & \vdots \\ \mathcal{M}_1(I \mp \Lambda_1)^K & \cdots & \mathcal{M}_K(I \mp \Lambda_K)^K & \boxed{O} \end{vmatrix}. \tag{3.13}$$

Now we consider the series which can also be written as a quasideterminant

$$\begin{aligned} \sum_{k=1}^{K}\mathcal{Q}_\pm[k]^{-1}\mathcal{Q}_\pm[k-1] &= \mathcal{Q}_\pm[1]^{-1}\mathcal{Q}_\pm[0] + \mathcal{Q}_\pm[2]^{-1}\mathcal{Q}_\pm[1] + \cdots \mathcal{Q}_\pm[K]^{-1}\mathcal{Q}_\pm[K-1] \\ &= -\begin{vmatrix} \mathcal{M}_1(I \mp \Lambda_1)^K & \cdots & \mathcal{M}_K(I \mp \Lambda_K)^K & O \\ \vdots & \cdots & \vdots & \vdots \\ \mathcal{M}_1(I \mp \Lambda_1)^2 & \cdots & \mathcal{M}_K(I \mp \Lambda_K)^2 & O \\ \mathcal{M}_1(I \mp \Lambda_1) & \cdots & \mathcal{M}_K(I \mp \Lambda_K) & I \\ \mathcal{M}_1 & \cdots & \mathcal{M}_K & \boxed{O} \end{vmatrix}. \end{aligned} \tag{3.14}$$

The above result can be readily proved by induction by using the properties of quasideterminants. Now the quasideterminant expression for the bosonic superfield conserved currents is therefore written as

$$
\begin{aligned}
D_\pm J_\pm[K+1] = {}& \begin{vmatrix} \mathcal{M}_1 & \cdots & \mathcal{M}_K & I \\ \mathcal{M}_1(I\mp\Lambda_1) & \cdots & \mathcal{M}_K(I\mp\Lambda_K) & O \\ \vdots & \cdots & \vdots & \vdots \\ \mathcal{M}_1(I\mp\Lambda_1)^K & \cdots & \mathcal{M}_K(I\mp\Lambda_K)^K & \boxed{O} \end{vmatrix} \\
& \left(D_\pm J_\pm - \left[\mathrm{i}J_\pm^2, \begin{vmatrix} \mathcal{M}_1(I\mp\Lambda_1)^K & \cdots & \mathcal{M}_K(I\mp\Lambda_K)^K & O \\ \vdots & \cdots & \vdots & \vdots \\ \mathcal{M}_1(I\mp\Lambda_1)^2 & \cdots & \mathcal{M}_K(I\mp\Lambda_K)^2 & O \\ \mathcal{M}_1(I\mp\Lambda_1) & \cdots & \mathcal{M}_K(I\mp\Lambda_K) & I \\ \mathcal{M}_1 & \cdots & \mathcal{M}_K & \boxed{O} \end{vmatrix} \right] \right) \\
& \times \begin{vmatrix} \mathcal{M}_1 & \cdots & \mathcal{M}_K & I \\ \mathcal{M}_1(I\mp\Lambda_1) & \cdots & \mathcal{M}_K(I\mp\Lambda_K) & O \\ \vdots & \cdots & \vdots & \vdots \\ \mathcal{M}_1(I\mp\Lambda_1)^K & \cdots & \mathcal{M}_K(I\mp\Lambda_K)^K & \boxed{O} \end{vmatrix}^{-1}.
\end{aligned}
$$

Similarly by using (3.13) in equation (3.11), we get

$$
\begin{aligned}
J_\pm^2[K+1] = {}& \begin{vmatrix} \mathcal{M}_1 & \cdots & \mathcal{M}_K & I \\ \mathcal{M}_1(I\mp\Lambda_1) & \cdots & \mathcal{M}_K(I\mp\Lambda_K) & O \\ \vdots & \cdots & \vdots & \vdots \\ \mathcal{M}_1(I\mp\Lambda_1)^K & \cdots & \mathcal{M}_K(I\mp\Lambda_K)^K & \boxed{O} \end{vmatrix} \\
& \times J_\pm^2 \begin{vmatrix} \mathcal{M}_1 & \cdots & \mathcal{M}_K & I \\ \mathcal{M}_1(I\mp\Lambda_1) & \cdots & \mathcal{M}_K(I\mp\Lambda_K) & O \\ \vdots & \cdots & \vdots & \vdots \\ \mathcal{M}_1(I\mp\Lambda_1)^K & \cdots & \mathcal{M}_K(I\mp\Lambda_K)^K & \boxed{O} \end{vmatrix}^{-1}.
\end{aligned}
$$

4. Relation with Mikhailov's superfield multisolitons

In this section, we relate the quasideterminant superfield mutisoliton solutions of the previous section with the solutions obtained by Mikhailov using an equivalent method known as the dressing method [2]. The superfield Darboux matrix (2.4) can also be written in terms of a superfield Hermitian projector. For this purpose we make use of equation (2.14), i.e. we write

$$
\begin{aligned}
\mathcal{S}|\Theta_i\rangle &= \lambda_i|\Theta_i\rangle, & i &= 1,2,\ldots,n \\
\mathcal{S}|\Theta_j\rangle &= \lambda_j|\Theta_j\rangle, & j &= n+1,n+2,\ldots,N.
\end{aligned}
$$

Now we set $\lambda_i = \mu$ and $\lambda_j = \bar{\mu}$, so that the matrix superfield $\mathcal{S}$ may be written as

$$\mathcal{S} = \mu\mathcal{P} + \bar{\mu}\mathcal{P}^\perp, \tag{4.1}$$

where $\mathcal{P}$ is the superfield Hermitian projector, i.e. $\mathcal{P}^\dagger = \mathcal{P}$. Also we have $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{P}^\perp = I - \mathcal{P}$. Now from (4.1)

$$\mathcal{S} = \mu\mathcal{P} + \bar{\mu}(I-\mathcal{P}) = (\mu-\bar{\mu})\mathcal{P} + \bar{\mu}I. \tag{4.2}$$

The superfield Darboux matrix is now expressed as

$$\mathcal{D}(\lambda) = \begin{vmatrix} \mathcal{M} & I \\ \mathcal{M}\Lambda & \boxed{\lambda I} \end{vmatrix} = (\lambda - \bar{\mu})\left(I - \frac{\mu - \bar{\mu}}{\lambda - \bar{\mu}}\mathcal{P}\right). \tag{4.3}$$

In expression (4.3) the Darboux-dressing function, expressed as quasideterminant containing the particular matrix solution $\mathcal{M}$ of the Lax pair (1.13), is shown to be expressed in terms of a superfield Hermitian projector $\mathcal{P}$ defined in terms of the particular column superfield solutions $|\Theta_i\rangle$ of the Lax pair. For the case of models based on Lie groups of $N \times N$ matrices, we set $\lambda_i = \mu$ $(i = 1, 2, \ldots, n)$ and $\lambda_j = \bar{\mu}$ $(j = n+1, \ldots, N)$, so that the solution $\mathcal{V}[2]$ is given by

$$\begin{aligned}\mathcal{V}[2] &= \left(\lambda I - \mu \sum_{i=1}^{n} \frac{|\Theta_i\rangle\langle\Theta_i|}{\langle\Theta_i|\Theta_i\rangle} - \bar{\mu} \sum_{j=n+1}^{N} \frac{|\Theta_j\rangle\langle\Theta_j|}{\langle\Theta_j|\Theta_j\rangle}\right)\mathcal{V} \\ &= (\lambda - \bar{\mu})\left(I - \frac{\mu - \bar{\mu}}{\lambda - \bar{\mu}} \sum_{i=1}^{n} \frac{|\Theta_i\rangle\,\langle\Theta_i|}{\langle\Theta_i|\,\Theta_i\rangle}\right)\mathcal{V} = (\lambda - \bar{\mu})\left(I - \frac{\mu - \bar{\mu}}{\lambda - \bar{\mu}}\mathcal{P}\right)\mathcal{V}.\end{aligned}$$

The Kth time iteration then gives the Kth superfield solution $\mathcal{V}[K+1]$ of the Lax pair

$$\begin{aligned}\mathcal{V}[K+1] &= \prod_{k=1}^{K} \begin{vmatrix} \mathcal{M}[K-k+1] & I \\ \mathcal{M}[K-k+1]\Lambda_{K-k+1} & \boxed{\lambda I} \end{vmatrix} \mathcal{V} \\ &= \prod_{k=1}^{K} (\lambda - \bar{\mu}_{K-k+1})\left(I - \frac{\mu_{K-k+1} - \bar{\mu}_{K-k+1}}{\lambda - \bar{\mu}_{K-k+1}}\mathcal{P}[K-k+1]\right)\mathcal{V},\end{aligned} \tag{4.4}$$

with the Kth superfield soliton solution $G[K+1]$ of the supersymmetric chiral model given by

$$G[K+1] = \prod_{k=1}^{K} (-\bar{\mu}_{K-k+1})\left(I + \frac{\mu_{K-k+1} - \bar{\mu}_{K-k+1}}{\bar{\mu}_{K-k+1}}\mathcal{P}[K-k+1]\right)\mathcal{V}, \tag{4.5}$$

where the superfield Hermitian projection in this case is

$$\mathcal{P}[k] = \sum_{i=1}^{n} \frac{|\Theta_i[k]\rangle\,\langle\Theta_i[k]|}{\langle\Theta_i[k]|\,\Theta_i[k]\rangle}, \qquad k = 1, 2, \ldots, K,$$

with

$$|\Theta_i[k]\rangle = \left(\lambda_i^{(k)} I - \mathcal{S}[k-1]\right)\left|\Theta_i^{(k)}\right\rangle,$$

and the kth particular matrix superfield solution $\mathcal{M}_k$ of the Lax pair is written in terms of kth particular column superfield solutions as

$$\mathcal{M}_k = \left(\left|\Theta_1^{(k)}\right\rangle, \left|\Theta_2^{(k)}\right\rangle, \ldots, \left|\Theta_N^{(k)}\right\rangle\right). \tag{4.6}$$

Now the expressions for the transformed superfield currents $J_\pm[K+1]$ are given by

$$J_\pm[K+1] = \prod_{k=1}^{K}\left(I \mp \frac{(\mu_{K-k+1} - \bar{\mu}_{K-k+1})}{(1 \mp \bar{\mu}_{K-k+1})}\mathcal{P}[K-k+1]\right) J_\pm \prod_{l=1}^{K}\left(I \mp \frac{(\bar{\mu}_l - \mu_l)}{(1 \mp \mu_l)}\mathcal{P}[l]\right). \tag{4.7}$$

Expressions (4.4), (4.5) and (4.7) can also be written as the sum of K terms by using the condition that $\mathcal{V}[K] = 0$ if $\lambda = \mu_i$, $\mathcal{V} = |\Theta_i\rangle$. The final expressions for $\mathcal{V}[K+1]$, $G[K+1]$ and $J_\pm[K+1]$ are then given by

$$\mathcal{V}[K+1] = \sum_{k,l=1}^{K} (\lambda - \bar{\mu}_k) \left(I - \frac{\mathcal{R}_k}{\lambda - \bar{\mu}_k} \right) \mathcal{V}, \tag{4.8}$$

$$G[K] = \sum_{k,l=1}^{K} (-\bar{\mu}_k) \left(I + \frac{\mathcal{R}_k}{\bar{\mu}_k} \right) G, \tag{4.9}$$

$$J_{\pm}[K] = \sum_{k=1}^{K} \left(I \mp \frac{\mathcal{R}_k}{1 \mp \bar{\mu}_k} \right) J_{\pm} \sum_{l=1}^{K} \left(I \mp \frac{\mathcal{R}_l}{1 \mp \mu_l} \right), \tag{4.10}$$

where the superfield function $\mathcal{R}_k$ is defined by

$$\mathcal{R}_k = \sum_{l=1}^{K} (\mu_l - \bar{\mu}_k) \sum_{i=1}^{n} \frac{\left|\Theta_i^{(k)}\right\rangle\!\left\langle\Theta_i^{(l)}\right|}{\left\langle\Theta_i^{(k)}\middle|\Theta_i^{(l)}\right\rangle}. \tag{4.11}$$

By expanding the right-hand side in equation (4.8), we see that the two expressions for the Kth iteration of $\mathcal{V}$, i.e. equations (3.4) and (4.8) are equivalent.

5. Darboux transformation on component fields

For the explicit soliton solutions of the supersymmetric chiral field, it is convenient to define the Darboux transformation on component fields of the superfields. For this purpose, we expand all superfields and collect different coefficients of θ's. In other words, the Darboux transformation on the component fields of the supersymmetric chiral field model can be defined by a Darboux matrix D:

$$D(\lambda) = \lambda I - S, \tag{5.1}$$

where D and S are the leading bosonic components of the matrix superfields $\mathcal{D}$ and $\mathcal{S}$, respectively. The matrix D is defined such that the solution V of the Lax pair (1.14) transforms as

$$\tilde{V} = D(\lambda)V, \tag{5.2}$$

with suitable conserved currents $\tilde{j}_{\pm}, \tilde{h}_{\pm}$ satisfying equations (1.8). That is $\tilde{V}$, $\tilde{j}_{\pm}, \tilde{h}_{\pm}$ are required to satisfy the transformed Lax pair

$$\partial_{\pm}\tilde{V} = \frac{1}{1 \mp \lambda}\tilde{j}_{\pm}\tilde{V} + \mathrm{i}\left(\frac{1}{1 \mp \lambda}\right)^2 \tilde{h}_{\pm}\tilde{V}. \tag{5.3}$$

Now following the arguments of the previous section, let

$$M = (V(\lambda_1)\,|1\rangle\,, \ldots, V(\lambda_N)\,|N\rangle) = (|m_1\rangle\,, \ldots, |m_N\rangle) \tag{5.4}$$

be an invertible $N \times N$ matrix, representing the leading bosonic component of the matrix superfield $\mathcal{M}$ with each column $|m_i\rangle = V(\lambda_i)\,|i\rangle$ in M representing a column solution of the Lax pair (1.14), when $\lambda = \lambda_i$, i.e.

$$\partial_{\pm} M = j_{\pm} M (I \mp \Lambda)^{-1} + \mathrm{i} h_{\pm} M ((I \mp \Lambda)^{-1})^2. \tag{5.5}$$

Note that the matrix S is now written in terms of Λ and M as

$$S = M\Lambda M^{-1}. \tag{5.6}$$

Thus, we have the Darboux transformation on the component fields $j_{\pm} \equiv j_{\pm}[1]$ and $\psi_{\pm} \equiv \psi_{\pm}[1]$ given by

$$(j_{\pm}, \psi_{\pm}, V) \longmapsto (\tilde{j}_{\pm}, \tilde{\psi}_{\pm}, \tilde{V}).$$

Again substituting (5.2) into equations (5.3) and using (1.14), we have

$$\tilde{j}_{\pm} = j_{\pm} \pm \partial_{\pm} S, \tag{5.7}$$

$$\tilde{\psi}_{\pm} = (I \mp S)\psi_{\pm}(I \mp S)^{-1}, \tag{5.8}$$

$$\tilde{h}_{\pm} = (I \mp S)h_{\pm}(I \mp S)^{-1}, \tag{5.9}$$

with the conditions on S given by

$$\partial_{+} S(I - S) = j_{+}S - Sj_{+} - \mathrm{i}h_{+} + \mathrm{i}(I - S)h_{+}(I - S)^{-1}, \tag{5.10}$$

$$\partial_{-} S(I + S) = -j_{-}S + S\,j_{-} + \mathrm{i}h_{-} - \mathrm{i}(I + S)h_{-}(I + S)^{-1}. \tag{5.11}$$

In fact $j_{\pm}$ and $h_{\pm}$ are the leading bosonic components of the matrix superfields $D_{\pm}J_{\pm}$ and $J_{\pm}^{2}$, respectively. Note that $\psi_{\pm}$ and $h_{\pm}$ transform exactly in the same manner. Therefore, once we write the expression for $\psi_{\pm}$, similar expression will hold for $h_{\pm}$. By using equations (5.10) and (5.11) in equation (5.7), we get

$$\begin{aligned}\tilde{j}_{\pm} &= (I \mp S)j_{\pm}(I \mp S)^{-1} - \mathrm{i}h_{\pm}(I \mp S)^{-1} + (I \mp S)\mathrm{i}h_{\pm}(I \mp S)^{-2}\\ &= (I \mp S)j_{\pm}(I \mp S)^{-1} - (I \mp S)[(I \mp S)^{-1}, \mathrm{i}h_{\pm}](I \mp S)^{-1}\\ &= Q_{\pm}[1](j_{\pm} + [\mathrm{i}h_{\pm}, Q_{\pm}[1]^{-1}])Q_{\pm}[1]^{-1},\end{aligned}$$

where by analogy of superspace calculations, we write

$$Q_{\pm}[i] = \prod_{i=1}^{k}(I \mp S[k]).$$

Since M and Λ are matrices, that do not commute, we can write equation (5.2) in terms of quasideterminants as

$$\begin{aligned}\tilde{V} &= D(\lambda)V = (\lambda I - S)\,V\\ &= (\lambda I - M\Lambda M^{-1})V = \begin{vmatrix} M & I \\ M\Lambda & \boxed{\lambda I}\end{vmatrix} V,\end{aligned} \tag{5.12}$$

and the chiral field $\tilde{g}$ is expressed as

$$\tilde{g} = \tilde{V}(0) = -Sg = -(M\Lambda M^{-1})g = \begin{vmatrix} M & I \\ M\Lambda & \boxed{O}\end{vmatrix} g. \tag{5.13}$$

Similarly the conserved currents $\tilde{j}_{\pm}$ and the spinor fields $\tilde{\psi}_{\pm}$ of the model are expressed as

$$\begin{aligned}\tilde{j}_{\pm} &= M(I \mp \Lambda)M^{-1}j_{\pm}M(I \mp \Lambda)^{-1}M^{-1} + M(I \mp \Lambda)M^{-1}\\ &\quad \times [\mathrm{i}h_{\pm}, M(I \mp \Lambda)^{-1}M^{-1}]M(I \mp \Lambda)^{-1}M^{-1}\\ &= \begin{vmatrix} M & I \\ M\,(I \mp \Lambda) & \boxed{O}\end{vmatrix} j_{\pm} \begin{vmatrix} M & I \\ M\,(I \mp \Lambda) & \boxed{O}\end{vmatrix}^{-1}\\ &\quad - \mathrm{i}\begin{vmatrix} M & I \\ M\,(I \mp \Lambda) & \boxed{O}\end{vmatrix}\left[h_{\pm}, \begin{vmatrix} M\,(I \mp \Lambda) & I \\ M & \boxed{O}\end{vmatrix}\right]\begin{vmatrix} M & I \\ M\,(I \mp \Lambda) & \boxed{O}\end{vmatrix}^{-1}\end{aligned} \tag{5.14}$$

and

$$\begin{aligned}\tilde{\psi}_{\pm} &= M(I \mp \Lambda)M^{-1}\psi_{\pm}M(I \mp \Lambda)^{-1}M^{-1}\\ &= \begin{vmatrix} M & I \\ M\,(I \mp \Lambda) & \boxed{O}\end{vmatrix} \psi_{\pm} \begin{vmatrix} M & I \\ M\,(I \mp \Lambda) & \boxed{O}\end{vmatrix}^{-1}.\end{aligned} \tag{5.15}$$

We can iterate the Darboux transformation K times and obtain the quasideterminant multisoliton solution of the supersymmetric model. For each $k = 1, 2, \ldots, K$, let M_k be an invertible $N \times N$ matrix solution of the Lax pair (1.14) at $\Lambda = \Lambda_k$; then the Kth solution $V[K+1]$ is expressed as

$$\begin{aligned}
V[K+1] &= \prod_{k=1}^{K} (\lambda I - S[K-k+1])\, V = \prod_{k=1}^{K} \begin{vmatrix} M[K-k+1] & I \\ M[K-k+1]\Lambda_{K-k+1} & \boxed{\lambda I} \end{vmatrix} V \\
&= \lambda V[K] - M[K]\Lambda_K M[K]^{-1} V[K] \\
&= \begin{vmatrix} M_1 & \cdots & M_K & I \\ M_1\Lambda_1 & \cdots & M_K\Lambda_K & \lambda I \\ \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^K & \cdots & M_K\Lambda_K^K & \boxed{\lambda^K I} \end{vmatrix} V.
\end{aligned} \tag{5.16}$$

The multisoliton solution $g[K+1]$ of the supersymmetric model can be readily obtained by taking $\lambda = 0$ in the expression of $V[K+1]$, i.e.

$$\begin{aligned}
g[K+1] &= \prod_{k=1}^{K} (-1)^k\, S[K-k+1] g = \prod_{k=1}^{K} \begin{vmatrix} M[K-k+1] & I \\ M[K-k+1]\Lambda_{K-k+1} & \boxed{O} \end{vmatrix} g \\
&= \begin{vmatrix} M_1 & M_2 & \cdots & M_K & I \\ M_1\Lambda_1 & M_2\Lambda_2 & \cdots & M_K\Lambda_K & O \\ M_1\Lambda_1^2 & M_2\Lambda_2^2 & \cdots & M_K\Lambda_K^2 & O \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ M_1\Lambda_1^K & M_2\Lambda_2^K & \cdots & M_K\Lambda_K^K & \boxed{O} \end{vmatrix} g.
\end{aligned} \tag{5.17}$$

Similarly the K times iteration of the Darboux transformation gives the following expression of the conserved currents:

$$\begin{aligned}
j_\pm[K+1] = &\begin{vmatrix} M_1 & \cdots & M_K & I \\ M_1(I \mp \Lambda_1) & \cdots & M_K(I \mp \Lambda_K) & O \\ \vdots & \cdots & \vdots & \vdots \\ M_1(I \mp \Lambda_1)^K & \cdots & M_K(I \mp \Lambda_K)^K & \boxed{O} \end{vmatrix} \\
&\left(j_\pm - \left[\mathrm{i}h_\pm, \begin{vmatrix} M_1(I \mp \Lambda_1)^K & \cdots & M_K(I \mp \Lambda_K)^K & O \\ \vdots & \cdots & \vdots & \vdots \\ M_1(I \mp \Lambda_1)^2 & \cdots & M_K(I \mp \Lambda_K)^2 & O \\ M_1(I \mp \Lambda_1) & \cdots & M_K(I \mp \Lambda_K) & I \\ M_1 & \cdots & M_K & \boxed{O} \end{vmatrix} \right] \right)
\end{aligned} \tag{5.18}$$

$$\times \begin{vmatrix} M_1 & \cdots & M_K & I \\ M_1(I \mp \Lambda_1) & \cdots & M_K(I \mp \Lambda_K) & O \\ \vdots & \cdots & \vdots & \vdots \\ M_1(I \mp \Lambda_1)^K & \cdots & M_K(I \mp \Lambda_K)^K & \boxed{O} \end{vmatrix}^{-1}. \tag{5.19}$$

For the spinor fields of the model, the K-fold Darboux transformation gives

$$\psi_\pm[K+1] = \begin{vmatrix} M_1 & \cdots & M_K & I \\ M_1(I \mp \Lambda_1) & \cdots & M_K(I \mp \Lambda_K) & O \\ \vdots & \cdots & \vdots & \vdots \\ M_1(I \mp \Lambda_1)^K & \cdots & M_K(I \mp \Lambda_K)^K & \boxed{O} \end{vmatrix}$$
$$\times \psi_\pm \begin{vmatrix} M_1 & \cdots & M_K & I \\ M_1(I \mp \Lambda_1) & \cdots & M_K(I \mp \Lambda_K) & O \\ \vdots & \cdots & \vdots & \vdots \\ M_1(I \mp \Lambda_1)^K & \cdots & M_K(I \mp \Lambda_K)^K & \boxed{O} \end{vmatrix}^{-1}. \tag{5.20}$$

6. Soliton solution of the $SU(2)$ model

For the construction of explicit solution in the matrix form using the Darboux transformation, we take the example of $\mathcal{G} = SU(2)$. The solutions can be obtained by the Darboux transformation by taking the trivial solution as the seed solution. We have been considering the case where $j_\pm, \mathrm{i}h_\pm \in su(2)$; the following discussion, however, are essentially the same for the Lie algebra $u(2)$. Let us take a most general unimodular 2×2 matrix representing an element of the Lie group $SU(2)$:

$$\begin{pmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{pmatrix},$$

where X and Y are complex numbers satisfying $X\bar{X} + Y\bar{Y} = 1$. Let $j_\pm, \mathrm{i}h_\pm = \mathrm{i}\psi_\pm^2$ are the non-zero constant (commuting) elements of $su(2)$, such that they are represented by anti-Hermitian 2×2 matrices:

$$\begin{aligned} j_+ &= \begin{pmatrix} \mathrm{i}p & 0 \\ 0 & -\mathrm{i}p \end{pmatrix}, \qquad & j_- &= \begin{pmatrix} \mathrm{i}q & 0 \\ 0 & -\mathrm{i}q \end{pmatrix}, \\ \mathrm{i}h_+ &= \begin{pmatrix} \mathrm{i}\alpha & 0 \\ 0 & -\mathrm{i}\alpha \end{pmatrix}, \qquad & \mathrm{i}h_- &= \begin{pmatrix} \mathrm{i}\beta & 0 \\ 0 & -\mathrm{i}\beta \end{pmatrix}, \end{aligned} \tag{6.1}$$

where p, q, α, β are non-zero real numbers. Since the bosonic field $\mathrm{i}h_\pm$ is a product of two Majorana spinors taking values in the Lie algebra $su(2)$, we can make a choice of matrix representation of $\mathrm{i}h_\pm$ as given in (6.1) identical to that of $j_\pm$ with real numbers α and β being products of two Grassmann numbers each satisfying the Majorana reality condition. The seed solution is then written as

$$g(x^+, x^-) = g[1] = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\{(p+\alpha)x^+ + (q+\beta)x^-\}} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}\{(p+\alpha)x^+ + (q+\beta)x^-\}} \end{pmatrix}. \tag{6.2}$$

The corresponding $V(\lambda)$ is

$$V(\lambda) = V[1](\lambda) = \begin{pmatrix} \omega(\lambda) & 0 \\ 0 & \omega^{-1}(\lambda) \end{pmatrix}, \tag{6.3}$$

where

$$\omega(\lambda) = \exp \mathrm{i}\left(\frac{1}{1-\lambda}px^+ + \frac{1}{1+\lambda}qx^-\right) \cdot \exp \mathrm{i}\left(\frac{1}{(1-\lambda)^2}\alpha x^+ + \frac{1}{(1+\lambda)^2}\beta x^-\right). \tag{6.4}$$

In this sense g, $j_\pm$, $h_\pm$ and V constitute the seed solution for the Darboux transformation.

Taking $\lambda_1 = \mu$ and $\lambda_2 = \bar{\mu}$, we have

$$\begin{aligned} M &= (V(\mu)|1\rangle, V(\bar{\mu})|2\rangle) = (|m_1\rangle, |m_2\rangle) \\ &= \begin{pmatrix} \omega(\mu) & \omega(\bar{\mu}) \\ -\omega^{-1}(\mu) & \omega^{-1}(\bar{\mu}) \end{pmatrix}. \end{aligned} \tag{6.5}$$

From (6.4), we see that

$$\overline{\omega}(\overline{\mu}) = \omega^{-1}(\mu), \qquad \omega(\mu) = \overline{\omega^{-1}}(\overline{\mu}). \tag{6.6}$$

By direct calculations, we note that

$$\det M = \mathrm{e}^{r+r'} + \mathrm{e}^{-(r+r')} > 0, \tag{6.7}$$

$$\begin{aligned} S &= M\Lambda M^{-1} \\ &= \frac{1}{\mathrm{e}^{r+r'} + \mathrm{e}^{-(r+r')}} \begin{pmatrix} \mu\, \mathrm{e}^{r+r'} + \bar{\mu}\mathrm{e}^{-(r+r')} & (\mu - \bar{\mu})\, \mathrm{e}^{\mathrm{i}(s+s')} \\ (\mu - \bar{\mu})\, \mathrm{e}^{-\mathrm{i}(s+s')} & \bar{\mu}\, \mathrm{e}^{r+r'} + \mu\, \mathrm{e}^{-(r+r')} \end{pmatrix}, \end{aligned} \tag{6.8}$$

where

$$\begin{aligned} r(x^+, x^-) &= \mathrm{i}\left(\frac{1}{(1-\mu)} - \frac{1}{(1-\bar{\mu})}\right) px^+ + \mathrm{i}\left(\frac{1}{(1+\mu)} - \frac{1}{(1+\bar{\mu})}\right) qx^-, \\ r'(x^+, x^-) &= \mathrm{i}\left(\frac{1}{(1-\mu)^2} - \frac{1}{(1-\bar{\mu})^2}\right) \alpha x^+ + \mathrm{i}\left(\frac{1}{(1+\mu)^2} - \frac{1}{(1+\bar{\mu})^2}\right) \beta x^-, \\ s(x^+, x^-) &= \left(\frac{1}{(1-\mu)} + \frac{1}{(1-\bar{\mu})}\right) px^+ + \left(\frac{1}{(1+\mu)} + \frac{1}{(1+\bar{\mu})}\right) qx^-, \\ s'(x^+, x^-) &= \left(\frac{1}{(1-\mu)^2} + \frac{1}{(1-\bar{\mu})^2}\right) \alpha x^+ + \left(\frac{1}{(1+\mu)^2} + \frac{1}{(1+\bar{\mu})^2}\right) \beta x^-, \end{aligned} \tag{6.9}$$

are linear functions of x^+ and x^-. The new solutions of the Lax pair and the field equations are

$$\tilde{V}(\lambda) = (\lambda I - S)\, V(\lambda), \qquad \tilde{g} = \tilde{V}|_{\lambda=0} \frac{1}{|\mu|}, \tag{6.10}$$

respectively. The above right-multiplier $\frac{1}{|\mu|}$ is used to keep $\tilde{g} \in SU\,(2)$. Now $\tilde{g}$ can be written as

$$\widetilde{g} = D(\lambda)|_{\lambda=0}\, g = -Sg, \tag{6.11}$$

$$= \begin{pmatrix} \widetilde{X} & \widetilde{Y} \\ -\overline{\widetilde{Y}} & \overline{\widetilde{X}} \end{pmatrix} g, \tag{6.12}$$

with

$$\widetilde{X} = \frac{\mu \mathrm{e}^{(r+r')} + \bar{\mu}\mathrm{e}^{-(r+r')}}{\mathrm{e}^{(r+r')} + \mathrm{e}^{-(r+r')}} \frac{1}{|\mu|}, \tag{6.13}$$

$$\widetilde{Y} = \frac{(\mu - \bar{\mu})}{\mathrm{e}^{(r+r')} + \mathrm{e}^{-(r+r')}} \mathrm{e}^{\mathrm{i}(s+s')} \frac{1}{|\mu|} \tag{6.14}$$

or

$$\widetilde{X} = \left[1 - \left(\frac{1}{2}\left|\frac{\mu - \bar{\mu}}{\mu}\right| \operatorname{sech}(r + r')\right)^2\right], \tag{6.15}$$

$$\widetilde{Y} = \frac{1}{2}\left|\frac{\mu-\bar{\mu}}{\mu}\right| \mathrm{sech}(r+r')\,\mathrm{e}^{\mathrm{i}(s+s')}. \tag{6.16}$$

The above expressions indicate that the functions $\widetilde{X}$ and $\widetilde{Y}$ have a solitonic form. Making the substitution $\mu = \mathrm{e}^{\mathrm{i}\theta}$ and simplifying, we get from (6.13) and (6.14)

$$\widetilde{X} = -(\cos\theta + \mathrm{i}\sin\theta\tanh(r+r')), \tag{6.17}$$

$$\widetilde{Y} = \mathrm{i}(\sin\theta\,\mathrm{sech}(r+r'))\,\mathrm{e}^{\mathrm{i}(s+s')}. \tag{6.18}$$

By using equations (6.1) and (6.8) in (5.7), we get the expressions for $j_\pm[1]$ and $h_\pm[1]$ as

$$\begin{aligned}
\tilde{j}_+ &= \begin{pmatrix} a & b \\ -\bar{b} & \bar{a}\end{pmatrix}, & \tilde{j}_-[1] &= \begin{pmatrix} c & d \\ -\bar{d} & \bar{c}\end{pmatrix},\\
\mathrm{i}\tilde{h}_+[1] &= \begin{pmatrix} \rho & \sigma \\ -\bar{\sigma} & \bar{\rho}\end{pmatrix}, & \mathrm{i}\tilde{h}_-[1] &= \begin{pmatrix} \gamma & \delta \\ -\bar{\delta} & \bar{\gamma}\end{pmatrix},
\end{aligned} \tag{6.19}$$

where

$$\begin{aligned}
a &= \mathrm{i}p\left[1 + \frac{1}{2}\frac{(\mu-\bar{\mu})^2}{|1-\mu|^2}\mathrm{sech}^2(r+r')\right] + \mathrm{i}\alpha\frac{1}{2}\frac{(\mu-\bar{\mu})^2}{|1-\mu|^2}\mathrm{sech}^2(r+r'),\\
b &= -\mathrm{i}p\left[-\frac{1}{2}\frac{(\mu-\bar{\mu})^2}{|1-\mu|^2}\tanh(r+r') + \frac{1}{2}(\mu-\bar{\mu})\right](\mathrm{sech}(r+r'))\,\mathrm{e}^{\mathrm{i}(s+s')}\\
&\quad - \mathrm{i}\alpha\left[-\frac{1}{2}\frac{(\mu-\bar{\mu})^2}{|1-\mu|^2}\tanh(r+r') - \frac{1}{2}\frac{(\mu^2-\bar{\mu}^2)}{(2-\mu-\bar{\mu})}\right](\mathrm{sech}(r+r'))\,\mathrm{e}^{\mathrm{i}(s+s')},\\
c &= \mathrm{i}q\left[1 + \frac{1}{2}\frac{(\mu-\bar{\mu})^2}{|1+\mu|^2}\mathrm{sech}^2(r+r')\right] + \mathrm{i}\beta\frac{1}{2}\frac{(\mu-\bar{\mu})^2}{|1+\mu|^2}\mathrm{sech}^2(r+r'),\\
d &= \mathrm{i}q\left[-\frac{1}{2}\frac{(\mu-\bar{\mu})^2}{|1+\mu|^2}\tanh(r+r') - \frac{1}{2}(\mu-\bar{\mu})\right](\mathrm{sech}(r+r'))\,\mathrm{e}^{\mathrm{i}(s+s')}\\
&\quad - \mathrm{i}\beta\left[-\frac{1}{2}\frac{(\mu-\bar{\mu})^2}{|1+\mu|^2}\tanh(r+r') + \frac{1}{2}\frac{(\mu^2-\bar{\mu}^2)}{(2-\mu-\bar{\mu})}\right](\mathrm{sech}(r+r'))\,\mathrm{e}^{\mathrm{i}(s+s')},\\
\rho &= \mathrm{i}\alpha\left[1 + \frac{1}{2}\frac{(\mu-\bar{\mu})^2}{|1-\mu|^2}\mathrm{sech}^2(r+r')\right],\\
\sigma &= \mathrm{i}\alpha\left[\left(1 + \frac{1}{4}\frac{(\mu-\bar{\mu})^2}{|1-\mu|^2}\mathrm{sech}^2(r+r')\right)^{\frac{1}{2}}\frac{(\mu-\bar{\mu})}{|1-\mu|}\mathrm{sech}(r+r')\,\mathrm{e}^{\mathrm{i}(s+s')}\right],\\
\gamma &= \mathrm{i}\beta\left[1 + \frac{1}{2}\frac{(\mu-\bar{\mu})^2}{|1+\mu|^2}\mathrm{sech}^2(r+r')\right],\\
\delta &= \mathrm{i}\beta\left[\left(1 + \frac{1}{4}\frac{(\mu-\bar{\mu})^2}{|1+\mu|^2}\mathrm{sech}^2(r+r')\right)^{\frac{1}{2}}\frac{(\mu-\bar{\mu})}{|1+\mu|}\mathrm{sech}(r+r')\,\mathrm{e}^{\mathrm{i}(s+s')}\right].
\end{aligned}$$

Taking $\mu = \mathrm{e}^{\mathrm{i}\theta}$, we may rewrite the above expressions as

$$\begin{aligned}
a &= \mathrm{i}p\left(1 - (1+\cos\theta)\mathrm{sech}^2(r+r')\right) - \mathrm{i}\alpha(1+\cos\theta)\mathrm{sech}^2(r+r'),\\
b &= -\mathrm{i}p\left[(1+\cos\theta)\tanh(r+r') + \mathrm{i}\sin\theta\right](\mathrm{sech}(r+r'))\,\mathrm{e}^{\mathrm{i}(s+s')}\\
&\quad - \mathrm{i}\alpha\left[(1+\cos\theta)\tanh(r+r') - \frac{\mathrm{i}\sin\theta\cos\theta}{1-\cos\theta}\right](\mathrm{sech}(r+r'))\,\mathrm{e}^{\mathrm{i}(s+s')},\\
c &= \mathrm{i}q(1 - (1-\cos\theta)\mathrm{sech}^2(r+r')) - \mathrm{i}\beta(1-\cos\theta)\mathrm{sech}^2(r+r'),
\end{aligned}$$

$$d = \mathrm{i}q\,[(1-\cos\theta)\tanh r - \mathrm{i}\sin\theta]\,(\mathrm{sech}(r+r'))\,\mathrm{e}^{\mathrm{i}(s+s')}$$
$$-\,\mathrm{i}\beta\left[(1-\cos\theta)\tanh(r+r') - \frac{\mathrm{i}\sin\theta\cos\theta}{1-\cos\theta}\right](\mathrm{sech}(r+r'))\,\mathrm{e}^{\mathrm{i}(s+s')},$$
$$\rho = \alpha\left(1-(1+\cos\theta)\mathrm{sech}^2(r+r')\right),$$
$$\sigma = \alpha\,[(1+\cos\theta)\tanh r + \mathrm{i}\sin\theta]\,(\mathrm{sech}(r+r'))\,\mathrm{e}^{\mathrm{i}(s+s')},$$
$$\gamma = \beta(1-(1-\cos\theta)\mathrm{sech}^2(r+r')),$$
$$\delta = \beta\,[(1-\cos\theta)\tanh r - \mathrm{i}\sin\theta]\,(\mathrm{sech}(r+r'))\,\mathrm{e}^{\mathrm{i}(s+s')}.$$

Equation (6.19) shows the novel result. We generate a new solution by starting from an arbitrary seed solution. In the asymptotic limit for $t \to \pm\infty, r, r' \to \pm\infty$ and the solution (6.12) reduces to the same expression as we obtained in the bosonic case [1]. Therefore these solutions are almost localized and behave asymptotically as a traveling wave. On comparison of the results obtained here with the bosonic case, we see that the soliton solution obtained for the supersymmetric chiral field model reduces to the results of the bosonic case [1] when the fermions are set to zero. Substituting above equations in (6.19), we get $\mathrm{Tr}\,\tilde{j}_+ = \mathrm{Tr}\,\tilde{h}_+ = \mathrm{Tr}\,\tilde{j}_- = \mathrm{Tr}\,\tilde{h}_- = 0$. As we have mentioned earlier, we also have $\mathrm{Tr}\,\partial_+ S = \mathrm{Tr}\,\partial_- S = 0$ from (5.10) and (5.11). Therefore $\tilde{j}_\pm$ and $\tilde{h}_\pm$ satisfy the additional constraints for $g \in SU(N)$. Consequently, when we use the above equations in (6.19), we get the explicit expressions of the conserved currents (solutions) $\tilde{j}_\pm$ and $\tilde{h}_\pm$ of the supersymmetric model by using the Darboux transformation.

7. Conclusions and outlook

In this paper, we have studied the supersymmetric principal chiral field model in two dimensions and obtained multisoliton solutions using the Darboux transformation. By starting with a superspace formalism, we expressed the Lax pair of the model in terms of component fields and expressed the solutions in terms of quasideterminants. We have applied the standard method of Darboux transformation to obtain the single as well as multisoliton solutions of the model. We have explicitly obtained the expressions of multisolitons for the $SU(2)$ model, and it reduces to the multisoliton solutions of the bosonic model when fermions are set equal to zero. Given the way in which such multisoliton solutions are constructed, it would be interesting to construct these solutions for other Lie groups and for more general supersymmetric nonlinear sigma models with target space being a compact symmetric space.

The main focus of our study in this paper has been the construction of classical multisoliton solutions of the supersymmetric chiral field model. An important aspect, however, is to investigate quantization of these solitons that has important physical significance regarding the quantum integrability and computation of exact quantities (e.g. exact factorizable S-matrix, mass-gap, etc) of the model as an integrable quantum field theory. It would also be worthwhile to study the binary Darboux transformation for the supersymmetric chiral model, by which the multisoliton solutions are obtained by defining a Hermitian projector expressed in terms of solutions of a (direct) Lax pair and an adjoint Lax pair.

Finally, there has been an increasing interest in the study of various integrable structures in string theory on $\mathrm{AdS}_5 \times S^5$ motivated by AdS/CFT correspondence, where the theory is regarded as a nonlinear sigma model with fields taking values in a supercoset space (see e.g. [22–27]). The bosonic strings on $\mathbb{R} \times S^3$, a submanifold of $\mathrm{AdS}_5 \times S^5$, are described in static gauge by an $SU(2)$ principal chiral model, the soliton solutions of which are used to construct classical string solutions. The principal chiral model we have studied has world-sheet supersymmetry, rather than the target space supersymmetry as is the case of string theory.

It would be interesting to consider principal chiral model with target space supersymmetry and to construct multisoliton solutions using the Darboux transformation and binary Darboux transformation. We hope to examine these problems in future.

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